Index of a family of lattice Dirac operators and its relation to the non-abelian anomaly on the lattice

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In the continuum, a topological obstruction to the vanishing of the non-abelian anomaly in 2n dimensions is given by the index of a certain Dirac operator in 2n+2 dimensions, or equivalently, the index of a 2-parameter family of Dirac operators in 2n dimensions. In this paper an analogous result is derived for chiral fermions on the lattice in the Overlap formulation. This involves deriving an Index Theorem for a family of lattice Dirac operators satisfying the Ginsparg–Wilson relation. The index density is proportional to Lüscher's topological field in 2n+2 dimensions.

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The Atiyah–Singer Index Theorem, both for single operators [1] and families of operators [2], has been of major importance in the modern development of Quantum Field Theory. For example, the Index Theorem for single operators gives topological insight into the non-vanishing of the axial anomaly [3], provides the basis for a resolution of the U(1) problem [4] and determines the dimension of the instanton moduli space (used in semiclassical investigations of Yang–Mills theory; see, e.g., [5]), while the Families Index Theorem reveals topological obstructions to the vanishing of gauge anomalies, thereby providing constraints on allowable theories [6–8].

When attempting to get a well-defined non-perturbative formulation of QFT's (in particular, gauge theories) one successful and well-established approach has been to formulate the theory on a spacetime lattice [9]. Therefore, carrying over the Index Theorem, both for single operators and families, to the lattice is an interesting and important problem. For a long time this did not seem possible though, due to the fermion doubling problem and the resulting need for acceptable lattice Dirac operators to break chiral symmetry [10]. In the traditional formulations, at best only a remnant of the Index Theorem is retained on the lattice [11]. However, the situation has changed quite dramatically in recent years with the advent of the Overlap formulation [12] and the discovery [13–15] of acceptable lattice Dirac operators satisfying the Ginsparg–Wilson relation [17]

$$D\gamma_5+\gamma_5D=aD\gamma_5D \qquad (a\text{=lattice spacing}) \tag{1}$$

Such operators have exactly chiral zero-modes (since $D\psi=0\ \Rightarrow\ D(\gamma_5\psi)=(aD\gamma_5D-\gamma_5D)\psi=0$), which allows to define $\mathrm{index}\,D\equiv\mathrm{Tr}(\gamma_5|_{\ker D})$ [15]. There is a "Lattice Index Theorem" [15,16,18]

$$\mathrm{index}\,D=-\frac{a}{2}\mathrm{Tr}(\gamma_5D)=a^4\sum_x q(x)\,. \tag{2}$$

where

$$q(x)=-\frac{a}{2}\mathrm{tr}(\gamma_5D(x,x)) \tag{3}$$

is the index density. For $SU(N)$ gauge fields on the Euclidean 2n-dimensional torus, $\mathrm{index}\,D$ and $q(x)$ reduce to the continuum index and density in the classical continuum limit [19], at least when D is the overlap Dirac operator [13]. (Earlier results in this direction were obtained in [15,18,20]. When D is the overlap Dirac operator the right-hand side of (2) has a spectral flow interpretation which had previously been used as a definition of lattice topological charge in [12].) Furthermore, although it is not invariant under the usual chiral transformations, the fermion action $S=a^4\sum_x\bar\psi(x)D\psi(x)$ exhibits an exact lattice-deformed version of chiral symmetry [16] (which was implicit in the overlap formalism): $\delta S=0$ for $\delta\psi=\hat{\gamma_5}\psi\,,\ \delta\bar\psi=\bar\psi\gamma_5$ where

$$\hat{\gamma_5}=\gamma_5(1-aD) \tag{4}$$

An easy consequence of (1) is ${\hat{\gamma_5}}^2=1$. Furthermore, after supplementing (1) with the γ_5-hermiticity condition

$$D^*=\gamma_5D\gamma_5 \tag{5}$$

we have ${\hat{\gamma_5}}^*=\hat{\gamma_5}$. Thus $\hat{\gamma_5}$ can be viewed as a lattice-deformed chirality matrix. The axial anomaly for the lattice-deformed chiral symmetry transformation above can be determined from the corresponding change in the fermion

measure to be $\mathcal{A}(x) = -ia\mathrm{tr}(\gamma_5 D(x,x)) = 2iq(x)$ [16]. This is completely analogous to the relation between axial anomaly and index density in the continuum [3].

Having seen that there is an exact Lattice Index Theorem for lattice Dirac operators satisfying the GW relation, and that the index and its density are related to the axial anomaly in precisely the same way as in the continuum, it is natural to ask if there is also a Lattice Index Theorem for *families* of such operators such that the families index is related to gauge anomalies (or more precisely, to obstructions to the vanishing of these anomalies) in the same way as in the continuum. In this paper we show that this is indeed the case: We derive an Index Theorem ((17) below) for a family of such lattice Dirac operators, parameterised by a 2-sphere in the orbit space of $SU(N)$ lattice gauge fields on the 2n-dimensional torus T^{2n}. This is the prototype for a more general Lattice Families Index Theorem which is currently under development [21]. We find that this index is related to an obstruction to gauge-invariance in precisely the same way as in the continuum setting, where it was previously studied by Alvarez-Gaumé and Ginsparg [6]. Furthermore, the index density is found to be proportional to Lüscher's topological field $q(x,y_1,y_2)$ in 2n+2 dimensions [23] (given by (26) below). This provides a natural origin for $q(x,y_1,y_2)$ in the lattice theory. (It was introduced in an ad hoc manner in [23].) This is of interest and potential use in connection with Lüscher's approach towards achieving gauge-invariance in nonabelian lattice chiral gauge theory: a local gauge anomaly-free formulation exists if and only if the local cohomology class represented by $q(x,y_2,y_2)$ is trivial [23].

In the continuum, the modulus of the chiral determinant $\det(i\not{D}_+^A)$ (suitably regularised as in [6]) is gauge-invariant, but anomalies may arise in the phase. Consider a family ϕ_θ of gauge transformations parameterised by $\theta \in S^1$. Each ϕ_θ is a map $T^{2n} \to SU(N)$ (we assume for simplicity that the fermion is in the fundamental representation). The action of ϕ_θ on A determines a circle-family $\{A^\theta\}_{\theta\in S^1}$ in the space of continuum $SU(N)$ gauge fields on T^{2n}. We restrict A to be in the topologically trivial sector (otherwise the chiral determinant vanishes). Then, generically, $\det(\not{D}_+^A) \neq 0$ and we have a map

$$S^1 \to S^1 \subset \mathbf{C} \quad , \qquad \theta \mapsto \det(i\not{D}_+^{A^\theta}) / \det(i\not{D}_+^A) \tag{6}$$

(where $S^1 \subset \mathbf{C}$ denotes the unit circle in $\mathbf{C}$). The winding number W_c of this map is an obstruction to gauge-invariance of the chiral determinant (since if the determinant is gauge-invariant then it is constant around the family $\{A^\theta\}_{\theta\in S^1}$ and the winding number is zero). In [6] the winding number W_c was shown to equal the index (which we define here to be the topological charge of the index bundle) of a 2-parameter family of Dirac operators $\not{D}^{(\theta,t)}$, or equivalently, the index of a Dirac operator $\mathcal{D}_c$ in 2n+2 dimensions, given as follows. (The subscript c here and in the following refers to "continuum".) The family A^θ is extended to a family $A^{(\theta,t)} = tA^\theta$ with $(\theta,t) \in B^2$, the unit disc. ($\{A^{(\theta,t)}\}$ corresponds to a 2-sphere in the orbit space of gauge fields since the $A^{(\theta,1)}$'s are all gauge-equivalent.) This determines the family of Dirac operators $\not{D}^{(\theta,t)} \equiv \not{D}^{A^{(\theta,t)}} = \gamma^\mu(\partial_\mu + A_\mu^{(\theta,t)})$. The Dirac operator $\mathcal{D}_c$ acting on spinor fields on $B^2 \times T^{2n}$ is now given by

$$\mathcal{D}_c = \Gamma^\alpha i(\partial_\alpha + \mathcal{A}_\alpha) \qquad\qquad \alpha = 1,\dots,2n+2 \tag{7}$$

where the gauge field $\mathcal{A}$ on $B^2 \times T^{2n}$ is given by $\mathcal{A}_\mu(\theta,t,x) = A_\mu^{(\theta,t)}(x)$, $\mathcal{A}_\alpha \equiv 0$ for $\alpha = 2n+1, 2n+2$, and the Dirac matrices in 2n+2 dimensions are chosen as $\Gamma^\mu = \sigma_1 \otimes \gamma^\mu$, $\Gamma^{2n+1} = \sigma_1 \otimes \gamma_5$, $\Gamma^{2n+2} = \sigma_2 \otimes 1$ where σ_j $(j=1,2,3)$ are the Pauli matrices and $\gamma_5 = i^n\gamma^1\gamma^2\cdots\gamma^{2n}$. The γ^μ's and σ_j's are taken to be hermitian so $\not{D}$ is anti-hermitian and $\mathcal{D}_c$ is hermitian. In (7) the derivatives for $\alpha = 2n+1, 2n+2$ are $\partial_{2n+1} = \frac{\partial}{\partial y_1}$, $\partial_{2n+2} = \frac{\partial}{\partial y_2}$ where (y_1,y_2) is a cartesian coordinate system on B^2. The polar coordinates θ and t are henceforth viewed as functions of (y_1,y_2). Let $\widetilde{B}^2$ denote another copy of the unit disc, then $B^2 \times T^{2n}$ and $\widetilde{B}^2 \times T^{2n}$ can be glued together along their common boundary $S^1 \times T^{2n}$ to get the closed manifold $S^2 \times T^{2n}$. An $SU(N)$ vectorbundle over this manifold is defined by taking the transition function on the common boundary $S^1 \times T^{2n}$ to be $\Phi(\theta,x)^{-1}$, where $\Phi(\theta,x) = \phi_\theta(x)$. The topological charge (Pontryargin number) of this bundle is then $-deg(\Phi)$, i.e. minus the degree (generalised winding number) of the map $\Phi : S^1 \times T^{2n} \to SU(N)$. The operator $\mathcal{D}_c$ above extends in a natural way to a Dirac operator (also denoted $\mathcal{D}_c$) on the spinor fields in this vectorbundle [6]. $\mathcal{D}_c$ anticommutes with the chirality matrix

$$\Gamma_5 \;\equiv\; i^{n+1}\Gamma^1\Gamma^1\cdots\Gamma^{2n+2} = \sigma_3 \otimes 1 \tag{8}$$

and thus has a well-defined index. The main result of [6] is that the obstruction to gauge-invariance discussed above is determined by this index:

$$W_c = -\mathrm{index}\,\mathcal{D}_c \tag{9}$$

The index of $\mathcal{D}_c$ can be calculated from the formula

$$\mathrm{index}\,\mathcal{D}_c = \mathrm{Tr}(\Gamma_5|_{\ker\mathcal{D}_c}) = \mathrm{Tr}(\Gamma_5\,e^{-\tau\mathcal{D}_c^2}) \quad \forall\tau>0 \tag{10}$$

This can be calculated in the $\tau \to 0$ limit by familiar techniques, leading to [6]

$$\mathrm{index}\,\mathcal{D}_c = -deg(\Phi) \tag{11}$$

Thus the obstruction is determined to be $W_c = deg(\Phi)$.

We now put a hypercubic lattice on T^{2n}, with lattice spacing a, and proceed to describe a lattice version of the preceding. Chiral gauge theory can be formulated on the lattice in the overlap formalism [12]. This approach can be reformulated as a functional integral approach with lattice Dirac operator D satisfying the GW relation (1) and γ_5-hermiticity condition (5) [22,23]. Let $\mathcal{C}$ denote the finite-dimensional space of lattice spinor fields on T^{2n}. The chiral projections $P_\pm = \frac{1}{2}(1\pm\gamma_5)$ and $\hat{P}_\pm = \frac{1}{2}(1\pm\hat{\gamma_5})$ determine decompositions $\mathcal{C}=\mathcal{C}_+\oplus\mathcal{C}_-$ and $\mathcal{C}=\hat{\mathcal{C}}_+\oplus\hat{\mathcal{C}}_-$ respectively. The (right-handed) lattice chiral determinant in this setting is

$$\det(iD_+^U) = \langle v_-\,,\hat{w}_+(U)\rangle \tag{12}$$

where v_- and $\hat{w}_+$ are unit volume elements on $\mathcal{C}_-$ and $\hat{\mathcal{C}}_+$ respectively. These are unique up to phase factors; they can be written as $v_- = v_1\wedge\cdots\wedge v_d$ and $\hat{w}_+ = \hat{w}_1\wedge\cdots\hat{w}_d$ where $v_1,\dots v_d$ and $\hat{w}_1,\dots,\hat{w}_d$ are orthonormal bases for C_- and $\hat{\mathcal{C}}_+$ respectively. v_- and $\hat{w}_+$ are the many-body groundstates in the overlap formulation [12], and correspond to the chiral fermion measures in the formulation of ref.'s [22,23]. Note that $\hat{\gamma_5}=\gamma_5(1-aD^U)$ depends on the lattice gauge field U, so the subspace $\hat{\mathcal{C}}_+$ and volume element $\hat{w}_+$ likewise depend on U. On the other hand, since the usual chiral decomposition $\mathcal{C}=\mathcal{C}_+\oplus\mathcal{C}_-$ does not involve U, neither does v_-. We are assuming $\dim\mathcal{C}_\pm = \dim\hat{\mathcal{C}}_\pm \equiv d$ (otherwise the chiral determinant vanishes). This is equivalent to assuming $\mathrm{index}\,D^U=0$, i.e. U is in the topologically trivial sector [12,23]. The space of lattice gauge fields will typically contain a subset of measure zero where D^U is not defined. In the case of the overlap Dirac operator such fields can be excluded by imposing a condition of the form $||1-U(p)||<\epsilon$ on the plaquette products of U [25]. This condition is automatically satisfied close to the classical continuum limit since $1-U(p)=a^2F_{\mu\nu}(x)+O(a^3)$. We will assume that the same is true for the general D that we are considering here.

Let $\{\phi_\theta\}_{\theta\in S^1}$ be a family of $SU(N)$ lattice gauge transformations, then the winding number W of the map

$$S^1\to S^1 \quad , \quad \theta\mapsto\langle v_-\,,\hat{w}_+(\phi_\theta\cdot U)\rangle/\langle v_-\,,w_+(U)\rangle \tag{13}$$

is an obstruction to gauge-invariance of the chiral determinant (12) just as in the continuum setting. This was recently studied in [24] where W was shown to reduce to W_c in the classical continuum limit. In the following we will show that this obstruction is related to the index of a family of lattice Dirac operators, defined as the index of a Dirac operator $\mathcal{D}$ in 2n+2 dimensions, in complete analogy with the continuum relation (9).

The action of ϕ_θ on U generates a circle-family $\{\phi_\theta\cdot U\}_{\theta\in S^1}$ in the space $\mathcal{U}$ of lattice gauge fields. Choose a disc-family $\mathcal{B}^2=\{U^{(\theta,t)}\}_{(\theta,t)\in B^2}$ in $\mathcal{U}$ such that $U^{(\theta,1)}=\phi_\theta\cdot U$. (Such a family might not exist in general due to the restrictions on $\mathcal{U}$ needed to ensure that D is well-defined. However its existence is guaranteed close to the classical continuum limit: we can take the lattice transcript of the continuum family $A^{(\theta,t)}$.) This determines a family of lattice Dirac operators $D^{(\theta,t)}=D^{U^{(\theta,t)}}$. Setting $\hat{\Gamma}^1=\sigma_1\otimes\hat{\gamma_5}$ (where $\hat{\gamma_5}=\hat{\gamma_5}^{(\theta,t)}$ is given by (4) with $D=D^{(\theta,t)}$) and $\hat{\Gamma}^2=\Gamma^2=\sigma_2\otimes 1$, we define the Dirac operator in 2n+2 dimensions in the lattice setting to be

$$\mathcal{D}=\hat{\Gamma}^\alpha i(\partial_\alpha+A_\alpha) \qquad \alpha=1,2 \tag{14}$$

The derivatives are with respect to the continuous cartesian coordinates (y_1,y_2) on B^2 and we have introduced a continuum SU(N) gauge field $A=A_\alpha dy_\alpha$ on B^2 with $A_\alpha(y_1,y_2,x)$ a function of lattice site x as well as (y_1,y_2). $\mathcal{D}$ extends in a natural way to an elliptic 1st order differential operator on the vectorfields with values in a vectorbundle over the closed manifold $S^2=B^2\cup_{S^1}\widetilde{B}^2$ as follows. The fibre of the vectorbundle is $\mathbf{C}^2\otimes\mathcal{C}$ (i.e. the representation space of the Pauli matrices tensored with the finite-dimensional vectorspace of lattice spinor fields on T^{2n}) and the transition function at the common boundary S^1 of B^2 and $\widetilde{B}^2$ is $1\otimes\Phi^{-1}$ where $\Phi(\theta)=\phi_\theta$. A vectorfield in this vectorbundle consists of a function $\Psi(\theta,t)$ on B^2 together with a function $\widetilde{\Psi}(\theta,s)$ on $\widetilde{B}^2$, both taking values in $\mathbf{C}^2\otimes\mathcal{C}$, and related at the common boundary S^1 by

$$\widetilde{\Psi}(\theta,1)=\Phi(\theta)^{-1}\cdot\Psi(\theta,1)\;\equiv\;1\otimes\phi_\theta^{-1}\cdot\Psi(\theta,1) \tag{15}$$

$\mathcal{D}$ is defined on Ψ by (14), and is defined to act on $\widetilde{\Psi}$ as $\hat{\Gamma}_U^\alpha i(\partial_\alpha+\tilde{A}_\alpha)$ where $\hat{\Gamma}_U^1=\sigma_1\otimes\hat{\gamma_5}^U$ and $\hat{\Gamma}_U^2=\Gamma^2$. The gauge-covariance of D implies that $D^{(\theta,1)}=D^{\phi_\theta\cdot U}=\phi_\theta\circ D^U\circ\phi_\theta^{-1}$ and $\hat{\gamma_5}^{(\theta,1)}=\phi_\theta\circ\hat{\gamma_5}^U\circ\phi_\theta^{-1}$. Using these it is

easily checked that $\mathcal{D}$ respects the relation (15), and is therefore a well-defined operator on the vectorfields in the above vectorbundle over S^2, provided the gauge field $\tilde{A} = \tilde{A}_\alpha dy_\alpha$ on $\widetilde{B}^2$ is related to the field A on B^2 at the common boundary S^1 by

$$\tilde{A}(\theta, 1, x) = \phi_\theta(x)^{-1} A(\theta, 1, x) \phi_\theta(x) + \phi_\theta(x)^{-1} d_\theta \phi_\theta(x) \tag{16}$$

(E.g. we can take $A \equiv 0$ and $\tilde{A}(\theta, s, x) = s\phi_\theta(x)^{-1} d_\theta \phi_\theta(x)$ in terms of polar coordinates (θ, s) on $\widetilde{B}^2$.)

The space of vectorfields in the above vectorbundle over S^2 is denoted by $\mathcal{V}$ in the following. The chirality operator $\Gamma_5 = \sigma_3 \otimes 1$ determines a chiral decomposition $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. The ellipticity of $\mathcal{D}$ follows easily from the facts that $\sigma_1 \otimes \hat{\gamma}_5$ anticommutes with $\sigma_2 \otimes 1$ and $(\sigma_1 \otimes \hat{\gamma}_5)^2 = (\sigma_2 \otimes 1)^2 = 1 \otimes 1$. Also, $\mathcal{D}$ is formally self-adjoint with respect to the natural inner product in $\mathcal{V}$ since $\sigma_1 \otimes \hat{\gamma}_5$ and $\sigma_2 \otimes 1$ are self-adjoint on $\mathbf{C}^2 \otimes \mathcal{C}$. $\mathcal{D}$ anticommutes with $\Gamma_5 = \sigma_3 \otimes 1$ and therefore has a chiral decomposition $\begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$ and $\mathrm{index}\, \mathcal{D} = \dim \ker \mathcal{D}_+ - \dim \ker \mathcal{D}_-$. The following formula for the index is derived below:

Theorem.

$$\mathrm{index}\, \mathcal{D} = -\frac{1}{2\pi i} \left(\int_{\mathcal{B}^2} \mathrm{Tr}(\hat{P}_+ d\hat{P}_+ d\hat{P}_+) + \frac{1}{2} \int_{\mathcal{S}^1} \mathrm{Tr}(\phi_\theta^{-1} d_\theta \phi_\theta {\hat{\gamma}_5}^U) \right) \tag{17}$$

Here $\hat{P}_+$ is to be viewed as a function on the space $\mathcal{U}$ of lattice gauge fields whose values are operators on $\mathcal{C}$ (i.e. finite-dimensional matrices), and d is the exterior derivative on $\mathcal{U}$. Thus the first integrand is a 2-form on $\mathcal{U}$ and can be integrated over the disc $\mathcal{B}^2$ in $\mathcal{U}$ to get a $\mathbf{C}$-number. The second integrand is a 1-form on the boundary $\mathcal{S}^1$ of $\mathcal{B}^2$, with ${\hat{\gamma}_5}^U = {\hat{\gamma}_5}^{(0,1)}$ constant.

By Eq.(3.11) of [24] the obstruction (winding number) W associated with the map (13) equals the right-hand side of (17) without the minus sign. It follows that

$$W = -\mathrm{index}\, \mathcal{D}\,. \tag{18}$$

This is the promised lattice analogue of (9). Since W reduces to $W_c = deg(\Phi)$ in the classical continuum limit [24], it follows that $\mathrm{index}\, \mathcal{D}$ reduces to $\mathrm{index}\, \mathcal{D}_c = -deg(\Phi)$ in this limit. The 2-form in the first term in the right-hand side of (17) has appeared previously in the overlap formalism in [26], where it was interpreted as a form of Berry's curvature. (The Berry phase is associated with the state $w_+(U)$ in (12).) It is interesting to note that a version of this 2-form also arises in the context of the quantised Hall effect [27]. The second term in (17) arises in [24] as the integral of the covariant gauge anomaly (and vanishes in the special case where $U = 1$).

The index formula (17), together with (25) for the index density, and the relation (18) are the main results of this paper. The proof is as follows. Analogously to (10) we have

$$\mathrm{index}\, \mathcal{D} = \mathrm{Tr}(\Gamma_5\, e^{-\tau \mathcal{D}^2}) \quad \forall \tau > 0\,. \tag{19}$$

This can be evaluated in the $\tau \to 0$ limit by the same familiar techniques used to evaluate (10) in [6]: It is seen to be the sum of a contribution from the B^2 part of $S^2 = B^2 \cup \widetilde{B}^2$, given by

$$\int_{B^2} d^2y\, a^4 \sum_x q_{\mathcal{D}}(x, y_1, y_2) \tag{20}$$

where

$$q_{\mathcal{D}}(x, y_1, y_2) = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2}\, \mathrm{tr}(\Gamma_5\, e^{-ik\cdot y} (e^{-\tau \mathcal{D}^2}) e^{ik\cdot y})(x, x)\,, \tag{21}$$

and an analogous contribution from the $\widetilde{B}^2$ part. In (21) the trace is over spinor and flavour indices; $\mathcal{O}(x, y)$ denotes the kernel function of an operator $\mathcal{O}$ on scalar lattice fields. Setting $\nabla_\alpha = \partial_\alpha + A_\alpha$ we find from (14) that

$$\begin{aligned} -\mathcal{D}^2 &= \hat{\Gamma}^\alpha \nabla_\alpha \hat{\Gamma}^\beta \nabla_\beta = \hat{\Gamma}^\alpha \hat{\Gamma}^\beta \nabla_\alpha \nabla_\beta + \hat{\Gamma}^\alpha [\nabla_\alpha\,, \hat{\Gamma}^1] \nabla_1 = \nabla_\alpha \nabla_\alpha + i\sigma_3 \hat{\gamma}_5 F_{12} + (\hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 - i\sigma_3 \nabla_2 \hat{\gamma}_5) \nabla_1 \\ &= \partial^2 + (2A_1 + \hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 - i\sigma_3 \nabla_2 \hat{\gamma}_5) \partial_1 + 2A_2 \partial_2 + \nabla_\alpha A_\alpha + i\sigma_3 \hat{\gamma}_5 F_{12} + (\hat{\gamma}_5 \nabla_1 \hat{\gamma}_5 - i\sigma_3 \nabla_2 \hat{\gamma}_5) A_1 \end{aligned} \tag{22}$$

where $\nabla_\alpha \hat{\gamma}_5 \equiv \partial_\alpha \hat{\gamma}_5 + [A_\alpha, \hat{\gamma}_5]$ (as in [23]); for notational simplicity we have omitted the $\otimes$ symbol. After substituting this in (20) and making a change of variables $k_j \to \tau^{-1/2} k_j$ we find

$$q_{\mathcal{D}}(x,y_1,y_2) = \lim_{\tau\to 0}\ \frac{1}{\tau}\int_{-\infty}^{\infty}\frac{d^2k}{(2\pi)^2}\,\mathrm{tr}(\sigma_3\ \exp\{-k^2+\sqrt{\tau}(2A_1+\hat{\gamma}_5\nabla_1\hat{\gamma}_5-i\sigma_3\nabla_2\hat{\gamma}_5)ik_1+\sqrt{\tau}(2A_2)ik_2 + \tau(\nabla_\alpha A_\alpha+\hat{\gamma}_5\nabla_1\hat{\gamma}_5+i\sigma_3(\hat{\gamma}_5F_{12}-\nabla_2\hat{\gamma}_5A_1))\})(x,x) \tag{23}$$

This can be calculated by expanding the integrand in powers of $\sqrt{\tau}$. Terms with odd powers of σ_3 give vanishing contribution, as do terms with k-dependence of the form $e^{-k^2}k_1^pk_2^q$ where either p or q is odd. The remaining terms are

$$\tau i e^{-k^2}\mathrm{tr}((2A_1\nabla_2\hat{\gamma}_5+\hat{\gamma}_5\nabla_1\hat{\gamma}_5\nabla_2\hat{\gamma}_5)k_1^2+\hat{\gamma}_5F_{12}-\nabla_2\hat{\gamma}_5A_1)(x,x)+O(\tau^{3/2}) \tag{24}$$

where we have used the fact that $\hat{\gamma}_5\nabla_\alpha\hat{\gamma}_5=-\nabla_\alpha\hat{\gamma}_5\hat{\gamma}_5$ (an easy consequence of $\hat{\gamma}_5{}^2=1$). Evaluating the integral in (23) with this integrand, we find that the contributions from the $2A_1\nabla_2\hat{\gamma}_5k_1^2$ and $-\nabla_2\hat{\gamma}_5A_1$ terms in $\mathrm{tr}(\cdots)$ in (24) cancel, resulting in

$$q_{\mathcal{D}}(x,y_1,y_2)=\frac{-1}{2\pi i}\Big(\frac{1}{4}\mathrm{tr}(\hat{\gamma}_5\nabla_1\hat{\gamma}_5\nabla_2\hat{\gamma}_5)(x,x)+\frac{1}{2}\mathrm{tr}(\hat{\gamma}_5F_{12})(x,x)\Big) \tag{25}$$

Modulo the numerical factor $-1/2\pi$, this coincides with Lüscher's topological field [23]

$$q(x,y_1,y_2)=-i\mathrm{tr}(\tfrac{1}{4}\hat{\gamma}_5[\nabla_1\hat{P}_-,\nabla_2\hat{P}_-]+\tfrac{1}{4}[\nabla_1\hat{P}_-,\nabla_2\hat{P}_-]\hat{\gamma}_5+\tfrac{1}{2}F_{12}\hat{\gamma}_5)(x,x)\,. \tag{26}$$

(Note that $\hat{\gamma}_5[\nabla_1\hat{P}_-,\nabla_2\hat{P}_-]=[\nabla_1\hat{P}_-,\nabla_2\hat{P}_-]\hat{\gamma}_5=\frac{1}{2}\hat{\gamma}_5\nabla_1\hat{\gamma}_5\nabla_2\hat{\gamma}_5$ (an easy consequence of $\hat{\gamma}_5\nabla_\alpha\hat{\gamma}_5=-\nabla_\alpha\hat{\gamma}_5\hat{\gamma}_5$) and that $\mathrm{tr}(\hat{\gamma}_5(x,x)F_{12}(x))=\mathrm{tr}(F_{12}(x)\hat{\gamma}_5(x,x))$.) Summing (25) over the lattice sites gives (cf. Appendix B of [23])

$$a^4\sum_x q_{\mathcal{D}}(x,y_1,y_2)=\frac{-1}{2\pi i}\mathrm{Tr}(\tfrac{1}{4}(\hat{\gamma}_5\partial_1\hat{\gamma}_5\partial_2\hat{\gamma}_5)-\tfrac{1}{2}\partial_1(A_2\hat{\gamma}_5)+\tfrac{1}{2}\partial_2(A_1\hat{\gamma}_5)) \tag{27}$$

The contribution to (20) from the first term in (27) gives the first term in the index formula (17). The contribution to (20) from the remaining terms in (27) reduces to $-\frac{1}{2}\int_{S^1}\mathrm{Tr}(A(\theta,1)\hat{\gamma}_5{}^{(\theta,1)})$ in polar coordinates. The analogous contribution to index $\mathcal{D}$ from the $\widetilde{B}^2$ part is only $+\frac{1}{2}\int_{S^1}\mathrm{Tr}(\tilde{A}(\theta,1)\hat{\gamma}_5{}^U)$ (since $\hat{\gamma}_5{}^U$ is constant). Adding these and using (16) we get the second term in (17).

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